Analysis of Wireless Information Systems Using MATLAB

Erfan Majeed

Sommersemester, 2014
General Information

- Every lecture will be subdivided into two parts
- The first part is devoted to Theory
  - general mathematics (stability, probability theory)
  - wireless communications
- The second part consists of programming exercises applying and illustrating the previously learned concepts and principles

Erfan Majeed
717a BB
erfan.majeed@kommunikationstechnik.org
Topics of this Lecture

- Numerical Analysis and Stability
- Probability Theory
- Modulation, Noise and Bit-Errors
- Radio Channels
- Equalization
- Diversity
- Code Division Multiple Access

- Final project which encompasses all covered Topics
Lecture 1: MATLAB and numerical Analysis
What is MATLAB?

- MATrix LABoratory
- Developed by The Mathworks, Inc (http://www.mathworks.com)
- Interactive, integrated, environment
  - for numerical computations
  - for symbolic computations
  - for scientific visualizations
- It is a high-level programming language
Features of MATLAB

- The MATLAB programming language
- Built-in MATLAB functions
- Graphics
- Linear algebra
- Data analysis
- Signal processing
- Polynomials and interpolation
- Numerically solving differential equations
- External interfaces: C, Fortran
- Linking C++ libraries (GSL, boost)
The Student Edition of MATLAB

• MATLAB Toolboxes
  ➢ Simulink
  ➢ Symbolic Math Toolbox

• Where Do I Get MATLAB? On-line Help?
  www.mathworks.com

• The latest version of MATLAB can be found at:

http://www.uni-due.de/zim/services/software/matlab_stud_installation.shtml
Characteristics of MATLAB

• Programming language based (principally) on matrices. Automatic memory management.

• This allows quick solutions to problems that can be formulated in vector or matrix form.

• Powerful GUI tools.

• Large collection of toolboxes: Many application-specific toolboxes available.
MATLAB Toolboxes

Math and Analysis
- Optimization
- Requirements Management Interface
- Statistics
- Neural Network
- Symbolic/Extended Math
- Partial Differential Equations
- PLS Toolbox
- Mapping
- Spline

Signal & Image Processing
- Signal Processing
- Image Processing
- Communications
- Frequency Domain System Identification
- Higher-Order Spectral Analysis
- System Identification
- Wavelet
- Filter Design

Control Design
- Control System
- Fuzzy Logic
- Robust Control
- μ-Analysis and Synthesis
- Model Predictive Control

Data Acquisition and Import
- Data Acquisition
- Instrument Control
- Excel Link
- Portable Graph Object

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MATLAB Data Types

• Basic type is the array
• Dimensioning is automatic
• Data objects:
  - integers
  - double
  - matrices
  - character (or text) strings
  - structures
  - cells
Display and Formatting

- MATLAB is case sensitive
- A semi-colon at the end of a line suppresses screen output
- To display one page at a time type (more on).

- Display formats:
  - `short`
  - `Long e, g, eng`
  - `hex`
  - `rat, bank`
Problems of Numerical Computations

- Floating point representation of real and rational numbers results in errors
- Defined by: IEEE Standard for Floating-Point Arithmetic (IEEE 754)
- Example: single-precision with 32 bits

\[ f = (-1)^{\text{sign}} \times \left( 1 + \sum_{n=1}^{23} b_n 2^{-n} \right) \times 2^{\text{exponent}-127} \]

- Definition machine epsilon $\varepsilon$: upper bound for the relative error of floating-point numbers
  - For single-precision: $\varepsilon = 2^{-24}$
  - Using double-precision floating-point numbers naturally increases the accuracy of the simulation
Effects of floating point errors

- Catastrophic cancellation
- Let’s assume we have two numbers $x$ and $x + \varepsilon$ who share the same floating-point representation $\tilde{x} = x$.
- The relative error for $x + \varepsilon$ is small:
  \[
  \frac{|x - (x + \varepsilon)|}{x + \varepsilon} = \frac{\varepsilon}{x} \approx 0
  \]
- However if we subtract $x + \varepsilon$ from $x$ the other the relative error becomes significant:
  \[
  \frac{|(\tilde{x} - \tilde{x}) - (x - (x + \varepsilon))|}{x - (x + \varepsilon)} = 1
  \]
- Having dramatic results for the following computation:
  \[
  \frac{1}{(x + \varepsilon) - x} \Rightarrow \frac{1}{\tilde{x} - \tilde{x}} \Rightarrow \frac{1}{0}
  \]
Catastrophic Cancellation – An Example

• The exponential function can be written as a Taylor-series

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

• and be approximated by truncating the series

\[ e^x \approx \sum_{n=0}^{N} \frac{x^n}{n!} \]

• for large negative values of \( x \) catastrophic cancellation leads to erroneous results (Demo Matlab)
Increasing Performance by Memoization

- Save previously computed values for later use
- Example: Compute a vector containing all factorials from 0! to N!

```matlab
res = zeros(1,N+1);
for j=0:N
    res(j+1) = factorial(j);
end
```

- The performance can be compared with the Matlab profiler

```
res = 0:N;
res(0) = 1;
for j=2:N
    res(j) = res(j)*res(j-1);
end
```

Profile Summary:

<table>
<thead>
<tr>
<th>Function Name</th>
<th>Calls</th>
<th>Total Time</th>
<th>Self Time*</th>
<th>Total Time Plot (dark band = self time)</th>
</tr>
</thead>
<tbody>
<tr>
<td>factorial</td>
<td>100001</td>
<td>1.953 s</td>
<td>1.953 s</td>
<td></td>
</tr>
<tr>
<td>factorial_vec1</td>
<td>1</td>
<td>2.310 s</td>
<td>0.357 s</td>
<td></td>
</tr>
<tr>
<td>factorial_vec2</td>
<td>1</td>
<td>0.016 s</td>
<td>0.016 s</td>
<td></td>
</tr>
</tbody>
</table>
Conditioning and Stability

• For a well-conditioned problem, small changes of the data result in small changes of the result
• This is not the case for an ill-conditioned problem
• However, even if a problem is well-conditioned, an unstable algorithm may produce erroneous results
• Example: Solving a linear system $Ax = b$ using Gauss elimination

$$
\begin{bmatrix}
\delta & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
\end{bmatrix}
$$

• With the solution:

$$
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= 
\frac{1}{1-\delta}
\begin{bmatrix}
-1 \\
1 \\
\end{bmatrix}
$$
Gaussian Elimination without Pivoting

\[
\begin{bmatrix}
\delta & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

• Gaussian Elimination yields:

\[
\begin{bmatrix}
\delta & 1 \\
0 & 1 - \frac{1}{\delta}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-\frac{1}{\delta}
\end{bmatrix}
\]

• If \( \varepsilon > 2 \times \delta \) this results in:

\[
\begin{bmatrix}
\delta & 1 \\
0 & -\frac{1}{\delta}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-\frac{1}{\delta}
\end{bmatrix}
\]

• Leading to:

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

• Really far from the truth
Gaussian Elimination with Pivoting

\[
\begin{bmatrix}
\delta & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 1 \\
\delta & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

• Gaussian Elimination yields:

\[
\begin{bmatrix}
1 & 1 \\
0 & 1 - \delta
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

• If \( \varepsilon > 2 \times \delta \) this results in:

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

• Leading to:

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
1
\end{bmatrix}
\]

• Which is much closer to the true result
Classifying Numerical Problems

- Vector norms:
  \[ \| v \|_k = \sqrt[k]{|v_1|^k + |v_2|^k + \cdots + |v_n|^k} \]

- Extension to matrix norms:
  \[ \| A \|_k = \max_{v \neq 0} \frac{\| Av \|_k}{\| v \|_k} \]

- Simpler formulas for \( k = 1, 2, \infty \):
  \[ \| A \|_1 = \max_j \sum_{i=1}^{m} |a_{i,j}| \]  
  Largest sum over a column

  \[ \| A \|_2 = \sqrt{\max_j \lambda_j (A^* A)} \]  
  \( \lambda_j \) = eigenvalue

  \[ \| A \|_{\infty} = \max_i \sum_{j=1}^{n} |a_{i,j}| \]  
  Largest sum over a row
The condition number

• How does a faulty matrix effect the outcome:

\[ Ax = b \quad (A + \Delta A)y = b \]

• If the following relation holds:

\[ \frac{\|\Delta A\|}{\|A\|} \leq \delta \]

• The relative error of the output is bounded by:

\[ \frac{\|x - y\|}{\|x\|} \leq \frac{\kappa(A)\delta}{1 - \kappa(A)\delta} \]

• Where \( \kappa(A) \) is the condition number of the matrix \( A \) defined by:

\[ \kappa(A) = \|A\| \|A^{-1}\| \]
Homework

• Write a matlab script that computes the Taylor series approximation (up to $n = 20$) of the exponential function, making use of memoization.

• Create a plot comparing the approximation and the true exponential function for $x \in [-10, 3]$.

• Given are two linear systems $Ax = b$, with $\delta = 0.002$:

  $\begin{bmatrix} \delta & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

  $\begin{bmatrix} 1 + \delta & \delta - 1 \\ \delta - 1 & 1 + \delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

• Calculate the exact solutions and the condition numbers.

• Plot all equations defined by the linear systems and relate them to the condition number.

• Consider the system $Ax = b + e$ where $e$ is a $2 \times 1$ vector which is normally distributed with a vanishing mean and a standard deviation of $\sigma = 0.0001$. Perform a Monte-Carlo experiment of 1000 runs. Create and explain plots of the errors for each system.
Analysis of Wireless Information Systems Using MATLAB

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Sommer Semester, 2014
Lecture 2: Random Variables and Channel Characteristics
Random Aspects in Communications

- How much of the transmitted power, on average, actually reaches the receivers?
- How does this power fluctuate?
- What are the effects of the noise that is being added by the system?
- What is the expected Bit-Error-Rate depending on the modulation scheme?
- How many people will use their cell-phone at the same time?

- Random behaviour can be described by random variables (RV)
  - Expected value $\mu = \text{mean}$
  - Standard Deviation $\sigma = \text{how strongly can the RV deviate from the mean}$
  - Discrete random variables
  - Continuous random variables
Mean, Standard Deviation, Variance

- Mean is the weighted average of all possible values that this random variable can take on.
- Variance shows how much variation there is from the average.
- Standard deviation is a measure of how far a set of numbers is spread out.
- MATLAB evaluate mean and variance using mean and var commands.

1. mean: Continuous: \( m_X = E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx \)
   Discrete: \( m_X = E\{X\} = \sum_{i \in R_X} x_i p(x_i) \)

2. variance: \( \sigma_X^2 = E\{(X - m_X)^2\} = E\{X^2\} - m_X^2 \)

3. Standard deviation: \( \sigma = \sqrt{\text{var}(X)} \)
Discrete Random Variables

- $X$ is a discrete random variable.
- The number of possible outcomes is finite or countable.

Example: $X$ is the outcome when a die is thrown.

- The possible values of $X = 1, 2, 3, 4, 5, 6$.
- The probability that $X$ takes the value $x_i = p(x_i) = P(X = x_i)$
- $p(x_i)$ measures the frequency with which event $x_i$ occurs.

- Some properties of discrete Random Variables:

\[
\sum_{i=1}^{N} p(x_i) = 1
\]

\[
\mu = \sum_{i=1}^{N} x_i p(x_i) \\
\sigma^2 = \sum_{i=1}^{N} (x_i - \mu)^2 p(x_i)
\]
Continuous Random Variables

• If the random variable can take values in a continuous interval (or a collection of intervals) – \( X \) = continuous random variable

• Characterized by the probability density function (pdf) \( f(x) \).

• Properties of the pdf:
  • Event probability: \( P(a \leq X < b) = \int_{a}^{b} f(x)dx \)
  • \( f(x) \geq 1 \quad \forall x \)
  • \( \int_{-\infty}^{\infty} f(x)dx = 1 \)

• Mean and Variances

\[
\mu = \int_{-\infty}^{\infty} x f(x)dx \quad \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx
\]
**Cumulative Distribution Function (CDF)**

Measures the probability that $X$ random variable has a value less or equal to $x$.

- **Discrete R.V.**
  
  $$F(x) = \sum_{i, x_i \leq x} p(x_i)$$

- **Continuous R.V.**
  
  $$F(x) = \int_{-\infty}^{x} f(t) dt$$

- **Properties of CDF function:**

  $$\lim_{x \to -\infty} F(x) = 0$$
  $$\lim_{x \to \infty} F(x) = 1$$
  $$a < b \implies F(a) \leq F(b)$$
**Histogram**

- A histogram is constructed by subdividing the interval \([a,b]\) containing a collection of data points into sub-intervals known as bins and count for each bin the number of data points that fall within that bin.

- The pdf estimated by finding the histogram and dividing the number of outcomes in each bin by number of realization.

- The function `hist` shows the histogram of sample values of a random variable.
Continuous Distributions

Uniform distribution

- This distribution is defined by interval $a$ and $b$, which are its minimum and maximum values.

$$p(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{ow} \end{cases}$$

- It is used for generating other types of distributions.

Ex: Generate a uniformly distributed random number between $[0,2]$.
ans. $A = 2 \times \text{rand}(1,1)$;
Ex: Generate $1e6$ uniformly distributed random numbers between $[0.5,2.5]$. 
The Normal / Gaussian Distribution

- Useful for modeling noise
  - Additive White Gaussian Noise (AWGN)
- $\mu$: expected value, mean
- $\sigma^2$: variance

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]
Generating Random Numbers

- In many cases we wish to generate random numbers that adhere to a given probability distribution
  - Many types of noise are normally distributed
  - Shot noise in a photodiode is Poisson distributed
  - The Rayleigh and Rice distributions are popular for modeling radio channels
- Computers usually transform uniformly distributed random numbers into samples with the desired distribution
- Generating good random numbers is difficult

“Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin.”
John von Neumann
Generating Random Numbers

• Luckily many applications do not require perfect random numbers

• A simple and reasonable solution is Lehmer’s Algorithm which generates a sequence $\mathcal{z}_n$ of random numbers:

$$\mathcal{z}_n = a \mathcal{z}_{n-1} \mod m$$

• The random numbers’ quality depends on the parameters $a$ and $m$

• $\mathcal{z}_n$ must never be 0, otherwise $\mathcal{z}_{n+1}$ is also 0, …
  • $m$ has to be prime

• Lehmer’s Algorithm is completely deterministic, for this reason it is often referred to as a pseudo-random number generator
**Inverse Transform**

- Let $X$ be a random variable with CDF $F_X(x)$.
- A random variable $U$ defined by $U = F_X(X)$ is uniformly distributed in the interval.
- Example: Normal Distribution

This also works the other way round:

$$X = F_X^{-1}(U)$$
Box-Muller Algorithm

- Generate a Gaussian distributed RV from 2 RVs $U_1$ and $U_2$ that uniformly distributed in the interval $0, 1$
- $X$ and $Y$ are two independent Gaussian distributed RVs
- Their joint probability density function is given by ($\sigma = 1, \mu = 0$)

$$f_{X,Y}(x, y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

- Radially symmetric $\rightarrow$ transform to polar coordinates

$$X = R \cos \Phi, \quad Y = R \sin \Phi$$

- $\Phi$ is iid, what about $R$?

$$P(R < r) = P(R < \sqrt{x^2 + y^2})$$
Box-Muller Algorithm, continued

\[ P(R < r) = \int \int f_{X,Y}(x, y) \, dx \, dy \]
\[ = \int_{0}^{r} \int_{0}^{2\pi} \frac{1}{2\pi} e^{-r^2/2} \, r \, d\phi \, dr \]
\[ = \int_{0}^{r} re^{-r^2/2} \, dr \]
\[ = 1 - e^{-r^2/2} \]

- Since naturally \( P(R < r_0) \in [0, 1] \) \( \forall r_0 \) we can compute \( R \) by the inverse transform

\[ 1 - e^{-R^2/2} = U \]

- Solving for \( R \) yields:

\[ R = \sqrt{-2 \ln(1 - U)} = \sqrt{-2 \ln(U_1)} \]
Box-Muller Algorithm, continued

- Obviously the relationship between $\Phi$ and $U_2$ is:
  \[ \Phi = 2\pi U_2 \]
- Putting everything together
  \[
  X = \sqrt{-2 \ln (U_1) \cos (2\pi U_2)} \\
  Y = \sqrt{-2 \ln (U_1) \sin (2\pi U_2)}
  \]
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Sommer Semester, 2014
Lecture 3: Quadrature Modulation and Non-Idealities of Hardware
Quadrature Modulation 101

- Let's assume we have the following signal: \( s(t) = A \cos(\omega t + \phi) \)
- Information can be encoded in:
  - The amplitude \( A \)
  - The frequency \( \omega \)
  - The phase \( \phi \)
- By using a trigonometric identity \( s(t) \) can be rewritten as
  - \( s(t) = A \left[ \cos(\omega t) \cos(\phi) - \sin(\omega t) \sin(\phi) \right] \)
- We will only use the amplitude and the phase to encode information
- Which motivates the following transmitter structure, where
  - Inphase = \( A \cos(\phi) \)
  - Quadrature = \( A \sin(\phi) \)
Quadrature Modulation 101: The Receiver

- We are examining the signal in the upper path.
- After the multiplication the signal $s_{rx}(t)$ is given by:
  $$s_{rx,U}(t) = \frac{A}{2} \left\{ [1 + \cos(2\omega t)]\cos(\phi) - [\sin(0) + \sin(2\omega t)]\sin(\phi) \right\}$$
- After lowpass filtering this simplifies to:
  $$s_{rx,U,LP}(t) = \frac{A}{2} \cos(\phi)$$
- The same thing can be done for the signal on the lower path.
**Complex Phasor Notation**

- The signal $s(t) = A \cos(\omega t + \phi)$ can also be written in complex phasor notation.
- Using Euler’s identity $e^{jx} = \cos(x) + j \sin(x)$.

\[
s(t) = \text{Re}\{Ae^{j\omega t + j\phi}\}
\]

- After some minor transformation this yields

\[
s(t) = \text{Re}\{e^{j\omega t}[A \cos(\phi) + jA\sin(\phi)]\} = \text{Re}\{e^{j\omega t}[I + jQ]\}
\]

- The Inphase and Quadrature components are the real and imaginary part of a complex number.
Quadrature Amplitude Modulation

- Modulation scheme which uses amplitude and phase to convey information
- Widely used in wireless communication as well as optics
- 4-QAM (QPSK), 16-QAM, 64-QAM, 256-QAM, …
- Higher order QAMs are more prone to noise

16-QAM (source Wikipedia)
When things go wrong ... Noise

- Noise
  - Amplifiers
  - Out-of-band noise
  - Other transmitters occupying the same band
- Usually modeled as complex Gaussian noise
When things go wrong ... Frequency Offset

- In many cases it is not possible to synchronize the oscillators at the receiver and the transmitter (frequency offset).
- Doppler shifts also affect the perceived frequency at the receiver.
- Solutions:
  - Atomic clocks, GPS disciplined oscillators.
  - GSM base station emit a reference signal which can be used by a cellphone to calibrate its oscillator.
  - Receiver can be designed so that they can compensate frequency shifts (Phase locked Loop, Costas loop).
When things go wrong ... Frequency Offset

- The mixers at the transmitter and the receiver have two different frequencies $f_1$ and $f_2$
- Let's analyze the signal in complex phasor notation
  
  $$s(t) = \text{Re}\{e^{j\omega_1 t}e^{-j\omega_2 t}[I + jQ]\}$$
  $$= \text{Re}\{e^{j(\omega_1-\omega_2)t}[I + jQ]\}$$

- The constellation diagram will rotate
When things go wrong ... IQ-Imbalance

- What happens to the constellation diagram if the two branches of the receiver are not perfect?

- The gains for the inphase and quadrature path may not be equal

- A phase shift might be present at the mixer
When things go wrong ... IQ-Imbalance

- Changing the gains at the receive and transmit path stretches and shrinks the constellation diagram
- Changing the phase at the mixer tilts and distorts the constellation diagram
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Lecture 4: Demodulating in the Presence of Noise
Noisy Signals

- A noisy signal can be considered to consist of two parts
  - The unperturbed signal $s$
  - The added noise $n$
- The received signal $r$ is then given as the superposition of $s$ and $n$

$$r = s + n$$

- Given the received signal $r$ can we draw any conclusion about the transmitted signal $s$?
- Usually we assume that the noise $n$ is Gaussian distributed
- We are interested in the probability distribution of $r$ if the signal $s$ was transmitted
- Which in the case of Gaussian noise is:

$$p(r|s) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(r-s)^2}{2\sigma^2}}$$
Demodulating BPSK

• Two symbols $s_1 = -A$ und $s_2 = A$
• Resulting in the following equations for the conditional PDFs

$$p(r|s_1) = \frac{1}{\sigma \sqrt{2 \pi}} e^{-\frac{(r+A)^2}{2\sigma^2}}$$

$$p(r|s_2) = \frac{1}{\sigma \sqrt{2 \pi}} e^{-\frac{(r-A)^2}{2\sigma^2}}$$

• Where to put the decision threshold $\xi$?
• Bit error Probabilities

$$P(e|s_1) = \int_{-\infty}^{\xi} p(r|s_1)dr = \frac{1}{\sigma \sqrt{2 \pi}} \int_{-\infty}^{\xi} e^{-\frac{(r+A)^2}{2\sigma^2}} dr$$

$$P(e|s_2) = \int_{-\infty}^{\xi} p(r|s_2)dr = \frac{1}{\sigma \sqrt{2 \pi}} \int_{-\infty}^{\xi} e^{-\frac{(r-A)^2}{2\sigma^2}} dr$$

• Total bit error probability

$$P(e) = \frac{1}{2} P(e|s_1) + \frac{1}{2} P(e|s_2)$$
Minimizing the Bit Error Probability

- Total bit error probability is equal to the shaded area

\[ P(e) = \frac{1}{2} P(e|s_1) + \frac{1}{2} P(e|s_2) \]

\[ p(r|s_1) \quad p(r|s_2) \]

- \( \xi = 0 \) yields the minimum bit error probability

\[ P(e) = \frac{1}{\sigma \sqrt{2\pi}} \int_0^\infty e^{-\frac{(r+A)^2}{2\sigma^2}} \, dr \]

\[ = 1 - \Phi \left( \frac{A}{\sigma} \right) \]

- Where \( \Phi(x) \) is the cumulative distribution function of a Gaussian normal distribution
Plotting BER curves

- Bit error probability depends on the amplitude and the noise variance
  \[ P(e) = 1 - \Phi \left( \frac{A}{\sigma} \right) \]

- Plotting the Bit Error Ratio versus the Signal to Noise or \( E_b/N_0 \) is a good starting point to analyze a system
  - Robustness to noise
  - Coding gain
  - Diversity
**Error Vector Magnitude**

- The Error Vector Magnitude is a figure of merit for distortions of the signal

\[
EVM = \sqrt{\frac{1}{N} \sum_{n=1}^{N} |E_{r,n} - E_{t,n}|^2} / |E_{t,m}|
\]

- Compares the received to the transmitted signal
- \(E_{t,m}\) is the constellation diagram vector with the maximum energy and is used for normalization purposes
- No demodulation of the signal is necessary
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Sommer Semester, 2014
Lecture 5: Radio Channels
The two path model

- A very simple model which covers interference

\[ d_1 = \sqrt{d^2 + (h_1 - h_2)^2} \quad d_2 = d_{2T} + d_{2R} = \sqrt{d^2 + (h_1 + h_2)^2} \]

- Electric farfield of an isotropic antenna

\[ E_T(r) = \sqrt{\frac{P_T G_T Z F_0}{2\pi}} \frac{e^{-jk_0r}}{r} \]

- The total electric field at the receive antenna is then given by

\[ E_T(r) = \sqrt{\frac{P_T G_T Z F_0}{2\pi}} \left( \frac{e^{-jk_0d_1}}{d_1} + \Gamma \frac{e^{-jk_0d_2}}{d_2} \right) \]
The two path model

• The following equation holds for the received Power

\[ P_R = G_R \frac{\lambda_0^2}{4\pi} \left| \frac{E_R(r)}{Z_{F0}} \right|^2 \]

• Inserting the E-field yields

\[ P_R = G_R G_T P_T \left( \frac{\lambda_0}{4\pi} \right)^2 \left| e^{-j\kappa_0 d_1} \frac{d_1}{d_1} + \Gamma e^{-j\kappa_0 d_2} \frac{d_2}{d_2} \right|^2 \]

• Some simplifying assumptions
  • \( d_1 \approx d_2 \approx d \) only valid for distance not for phase
  • Perfectly conducting plane, horizontal polarization: \( \Gamma = -1 \)
  • \( h_1 \ll d, h_2 \ll d \)

\[ P_R = G_R G_T P_T \left( \frac{\lambda_0}{4\pi d} \right)^2 \sin^2 \left( \frac{k_0 h_1 h_2}{d} \right) \]
Two Path Model

- Path loss for $h_1 = 30m$, $h_2 = 1.5m$, $f = 900MHz$, horizontal polarization
Rayleigh Distribution

- Useful for simulating radio channels with multipath propagation
- Assumptions:
  - The signal at the receiver consists of multiple copies of the transmitted signal
  - The phases of the various multipath components MPC are uniformly and iid distributed on the interval $[0, 2\pi]$
  - The amplitudes of the MPCs are of the same magnitude
- These assumption are roughly given in an urban scenario with no line of sight
- The signal $r(t)$ at the receiver is then given by the superposition of the various MPCs

$$r(t) = \sum_{n=1}^{N} a_n \cos(\omega t + \varphi_n)$$
Rayleigh Distribution

- $r(t)$ can also be written in complex phasor notation and with inphase and quadrature components

$$r(t) = \Re \left\{ \sum_{n=1}^{N} [I_N + jQ_N] e^{-j\omega t} \right\}$$

- $I_N$ and $Q_N$ are uncorrelated
- Their sums are Gaussian distributed (Central Limit Theorem)
- What are the probability density functions of phase and amplitude?
- The amplitude is Rayleigh distributed

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$
Doppler Spectrum

- What frequency shifts can be expected when the receiver is moving?
- Doppler frequencies depend on the angles of the impinging signals

\[ f_D = g(\alpha) = f_{D,\text{max}} \cos(\alpha) \]

- How does the random distribution of the angles translates to the random distribution of the frequencies?
- Two angles result in the same Doppler shift
- The slope of \( g(\alpha) \) is also important

\[ p_{f_D}(f) = \sum_{n=1}^{N} p_{\alpha}(g^{-1}(f)) \left| \frac{d}{df} g^{-1}(f) \right| \]
Jakes Power Spectrum Density

- The angles of arrival are uniformly distributed in the interval $0, 2\pi$

$$f_D = g(\alpha) = f_{D,\text{max}} \cos(\alpha)$$

- With $\frac{d}{dx} \arccos(x) = \frac{-1}{1-x^2}$ this results in:

$$p f_D(f) = \frac{1}{\pi f_{D,\text{max}} \sqrt{1 - \left(\frac{f}{f_{D,\text{max}}}\right)^2}}$$

- Assuming ideal isotropic scattering, i.e. all amplitudes of the various multipaths are identical, yields for the resulting PSD

$$S(f) \sim p f_D(f)$$
Jakes Power Spectrum Density, continued

- With

\[
\int_{-\infty}^{\infty} S(f) \, df = 2\sigma_0^2 \quad \int_{-\infty}^{\infty} p_{f_D}(f) \, df = 1
\]

- We obtain the following equation for the Doppler spectrum

\[
S(f) = \begin{cases} 
\frac{2\sigma_0^2}{\pi f_{D,\text{max}}} \sqrt{1 - \left(\frac{f}{f_{D,\text{max}}}\right)^2} & \text{for } f < f_{D,\text{max}} \\
0 & \text{otherwise}
\end{cases}
\]
**Channel Impulse Response**

- The channel impulse response $h$ gives the relationship between the input $x$ and the output signal $y = h \ast x$
- Simple example: 2 path model
- If a radio channel varies with time, the channel impulse response is a two dimensional matrix $h(t, \tau)$
  - $\tau$ for the delay
  - $t$ for the time
**Channel Sounding**

- Measuring the channel impulse response
- We need a transmitter and a receiver

There exist several kinds of channel sounders
- Impulse
- Correlation
- Frequency domain

This setup uses a correlation channel sounder
Measurements

- Measurements were conducted outdoors
- The receive antenna was either moved or kept stationary
- Fading is evident in the case of a moving receive antenna

KIT Campus (source Google maps)
The Power Delay Profile

- Shows how the power is distributed with respect to the delay $\tau$
- It can be computed for the channel impulse response
  \[
PDP(\tau) = E\{|h(\tau)|^2\}
  \]
- If the channel is assumed to be stationary it can also be computed by taking the average over several samples
  \[
PDP(\tau) = \int_{-\infty}^{\infty} |h(t, \tau)|^2 dt
  \]
- Comparison of the measurements with the COST 207 rural area model
Figure of Merit of the Power Delay Profile

- Average delay
  \[
  \tau_d = \frac{\int_{-\infty}^{\infty} \tau PDP(\tau) d\tau}{\int_{-\infty}^{\infty} PDP(\tau) d\tau}
  \]

- Delay spread
  \[
  \sigma_d = \sqrt{\frac{\int_{-\infty}^{\infty} (\tau - \tau_d)^2 PDP(\tau) d\tau}{\int_{-\infty}^{\infty} PDP(\tau) d\tau}}
  \]

- The delay spread shows how much the power of the PDP is spread (compare with variance of a random variable)

- If the delay spread is larger than symbol duration inter symbol interference will occur

- Analysis can also be done in frequency domain
Delay Spread and Coherence Bandwidth

- The inverse of the delay spread is proportional to the coherence Bandwidth $B_C$
  \[
  B_C \approx \frac{1}{\sigma_d}
  \]
- If the bandwidth of the transmitted signal is smaller than the coherence bandwidth no equalization is necessary at the receiver  
  $\Rightarrow$ frequency flat channel
- If the bandwidth is larger $\Rightarrow$ frequency selective channel and an equalizer is necessary
Scattering Function

- The scattering function shows how the power is distributed in the delay domain and the frequency/Doppler domain

\[ S_S(f, \tau) = \left| \int_{-\infty}^{\infty} h(t, \tau)e^{-j2\pi ft} dt \right|^2 \]

- Comparison of a stationary and mobile receive antenna

stationary

mobile
The Doppler Spectrum

• The Doppler spectrum can be computed from the scattering function

\[ S(f) = \int_{0}^{\infty} S_S(f, \tau) d\tau \]

• Average Doppler shift

\[ \bar{f}_D = \frac{\int_{-\infty}^{\infty} f_D S(f_D) df_D}{\int_{-\infty}^{\infty} S(f_D) df_D} \]

• Doppler spread

\[ \sigma_{f_D} = \sqrt{\frac{\int_{-\infty}^{\infty} (f_D - \bar{f}_D)^2 S(f_D) df_D}{\int_{-\infty}^{\infty} S(f_D) df_D}} \]

• Comparing the Doppler spread with the symbol period one differentiates between slow and fast fading
Analysis of Wireless Information Systems Using MATLAB

Erfan Majeed

Sommer Semester, 2014
Lecture 6: Diversity
Basic Principle of Diversity

- How can a receiver exploit the changing nature of a radio channel?
- There exist several kinds of diversity:
  - Temporal diversity
  - Frequency diversity
  - Angular diversity
  - Spatial diversity
  - Polarization diversity
- Everyone of these diversity types takes advantage of different characteristics of the radio channel.
Temporal and Frequency diversity

- A radio channel can be considered constant over its coherence time $T_C$.
- If a signal is repeated after $T_C$, it will benefit from diversity.
- A more sophisticated version is a combination of interleaving and coding which effectively spread a signal in time domain.
- In a static scenario there is no temporal diversity!

- The coherence bandwidth $B_C$ of a radio channel is responsible for frequency diversity.
- Frequencies which are spaced more apart than $B_C$ can be considered as uncorrelated.
- Transmit the same signal at two frequencies.
- Use spread spectrum techniques
  - CDMA
  - Frequency hopping
Angular Diversity

• Angular diversity exploits the fact that multipath components usually arrive from different angles
• Usually an antenna array has to be employed
• Depending on the antenna pattern certain MPCs can be filtered out in order to suppress interference

• The angular spread has to be large enough for angular diversity
• For the uplink to a base station (BS) this is seldom the case, as most of the scatterers are near the mobile station (MS)
Spatial and Polarization diversity

• The received signals at antennas which are placed at different locations are usually uncorrelated.

• The distance has to be in the order or greater than the wavelength of the received signals.

• The angular spread also has an effect on the interference pattern and therefore on the necessary minimal distance to obtain uncorrelation.

• Polarization diversity exploits the fact that horizontal and vertical polarization are affected differently by the radio channel.

• Since the angular spread for the MS to BS channel is usually low, many base stations use polarization diversity since it requires less space than spatial diversity.
Combination of Signals

- The easiest example is spatial diversity with multiple antennas
- For other kinds of diversity there exist similar methods
- Two kinds of combination techniques
  - Selection diversity
  - Combining diversity

Selection diversity
- Select the strongest signal
- Switch if the current signal drops below a certain threshold

Combining diversity
- Equal Gain Combining: Add all received signals with equal weighing
- Maximum Ratio Combining: weight the received signals by the complex conjugated of the channel coefficient $h_{m,n}$
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Sommer Semester, 2014
Lecture 7: Equalization
Coping with Frequency Selective Channels

• A channel can be described by its channel impulse response
• Intersymbol Interference (ISI) will occur if the channels delay spread is larger than the symbol period
• ISI leads to a worse performance of the system
• Equalizers can mitigate the effects of ISI
• The sampled received signal can be modeled as

\[ r(t) = \sum_{n=-\infty}^{\infty} I(n) h(t - nT) + n(t) \]

- Where \( T \) is the symbol period, \( I(n) \) are the transmitted symbols and \( h(t) \) is the channel response to an input symbol’s pulse
- Matched Filtering and sampling with period \( T \) yields

\[ y(n) = \int_{-\infty}^{\infty} r(t) h^*(t - nT) dt \]
Estimating the transmitted Symbols

- How can we estimate $I(n)$ from $y(n)$?
- Maximum Likelihood Sequence Estimation or linear or non-linear equalizers
- For a linear equalizer the output signal is given by

$$\hat{I}(k) = \sum_{j=-K}^{K} c(j)y(k - j)$$

- The cascade of the channel, the matched filter and the equalizer can be written as one single filter

$$q(m) = \sum_{j=-K}^{K} c(j)x(m - j)$$

- Where $x(m)$ is the response of the matched filter to $h(t)$
- Here we don’t use noise whitening!
The estimate of $I(n)$ is then given by

$$\hat{I}(k) = q_0 I(k) + \sum_{n \neq k} I(n) q(k - n) + \sum_{j=-K}^{K} c(j) \hat{n}(k - j)$$

The first part is the wanted signal, the second part is ISI and the last part noise.

A very simple scheme to eliminate ISI is **Zero Forcing** where the equalizer coefficients are picked in such a way that:

$$q(m) = \begin{cases} 1 & (m = 0) \\ 0 & (m \neq 0) \end{cases}$$

This method has the serious drawback that it completely ignores noise for computing the equalizer coefficients.
Convolution with a Toeplitz-matrix

- A matrix $A$ is a Toeplitz matrix if its elements $a_{i,j}$ are a function of the row and column indexes' differences $(i - j)$

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{pmatrix}
= 
\begin{pmatrix}
h_0 & h_{-1} & h_{-2} \\
h_1 & h_0 & h_{-1} \\
h_2 & h_1 & h_0
\end{pmatrix}
\]

- The discrete convolution is defined as

\[w(i) = \sum_{j} u(j)v(i - j)\]

- But it can also be written as a matrix multiplication with a Toeplitz matrix

\[
\begin{pmatrix}
v_0 & v_{-1} & 0 \\
v_1 & v_0 & v_{-1} \\
0 & v_1 & v_0
\end{pmatrix}
\begin{pmatrix}
u_{-1} \\
u_0 \\
u_1
\end{pmatrix}
= 
\begin{pmatrix}
w_{-1} \\
w_0 \\
w_1
\end{pmatrix}
\]
Calculating the Zero Forcing coefficients

• We want to pick the filter coefficients in such a way, that:

\[ q(m) = \begin{cases} 
1 & (m = 0) \\
0 & (m \neq 0) 
\end{cases} \quad \text{with} \quad q(m) = \sum_{j=-K}^{K} c(j) x(m - j) \]

• Creating a Toeplitz matrix results in

\[
\begin{pmatrix}
 x_0 & x_{-1} & 0 \\
 x_1 & x_0 & x_{-1} \\
 0 & x_1 & x_0 
\end{pmatrix}
\begin{pmatrix}
 c_{-1} \\
 c_0 \\
 c_1 
\end{pmatrix}
= \begin{pmatrix}
 0 \\
 1 \\
 0 
\end{pmatrix}
\]

• Which can be solved easily for the equalizer coefficients
Limits of linear Equalizers

• Assuming we have the following channel
  \[ h(t) = \delta(t) + \alpha \delta(t - \tau) \]

• The Z-Transform is given by
  \[ H(z) = 1 + \alpha z^{-1} \]

• A perfect Equalizer would be
  \[ H_{eq}(z) = \frac{1}{1 + \alpha z^{-1}} \]

• Which can be written using a geometric series as
  \[ H_{eq}(z) = \sum_{k=0}^{\infty} (\alpha z^{-1})^k = 1 + \alpha z^{-1} + \alpha^2 z^{-2} \ldots \]

• An infinitely long equalizer is necessary
The Eye Diagram

• The eye diagram is generated by overlaying the pulses of subsequent symbols.

• For many digital modulation schemes the resulting diagram looks like an eye; Hence the name eye diagram.

• Eye diagrams highlight several distortions of the signal such as
  • Noise
  • Jitter
  • Intersymbol Interference
Analysis of Wireless Information Systems Using MATLAB

Erfan Majeed

Sommer Semester, 2014
Lecture 8: Spread Spectrum
Spread Spectrum

- The narrowband information-bearing signal is spread over a larger bandwidth
  - Suppression of Interference
  - Multiple Access
  - Concealing the Signal
- Many ways to spread the signal
- There exist also hybrid forms such as FH/DS
**Direct Sequence Spread Spectrum**

- Multiply the data with a pseudonoise sequence
- The data stream’s symbol period $T_s$ is much shorter than the PN sequence’s chip duration $T_c$

A multiplication in time domain results in a convolution in frequency domain.
Rake Receiver

- The receiver has to correlate the received signal with the same pseudonoise sequence which was used in the transmitter.
- In a multipath environment we will get multiple peaks, each one corresponding to one distinct MPC and containing useful information.
- The Rake Receiver exploits this diversity by processing every MPC separately.

Different combining schemes can be used.
- However the Rake receiver has to have channel state information.
Orthogonal Signals

• Just as for vectors we can also compute the **inner product** of two continuous functions

\[
\langle f_1(t), f_2(t) \rangle = \int_a^b f_1(t)f_2^*(t)dt
\]

• If the inner product is zero \( f_1(t) \) and \( f_2(t) \) are called orthogonal

• We can also define a **norm** for continuous functions

\[
\|f(t)\| = \left( \int_a^b |f(t)|^2 dt \right)^{1/2}
\]

• A set of functions is called orthonormal if their inner products are zero and their norms equal to unity
Advantages of DSSS

- Suppression of Interference since narrowband jammers are spread at the receiver

- Code Division Multiple Access, if the spreading sequences are orthogonal

- A signal can be concealed if its power is below the noise power
Important Properties of Spreading Codes

- Perfect Autocorrelation ($\equiv \delta(t)$) facilitates synchronization and prevents interchip interference
- Spreading Codes should be orthogonal for multiple access schemes
- Easier to realize in Downlink than in Uplink
- Low cross correlation is mandatory for the Uplink
  - The codes of the various users are not synchronized due to the different delays
  - Power control is needed
- For multiple access scheme a large number of spreading codes should be available
Generation of Spreading Codes

- Walsh-Hadamard codes
  - Defined recursively by a matrix
    \[
    H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
    \]
    \[
    H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}
    \]
  - The columns/rows are orthogonal

- Pseudonoise (PN) sequences
  - Generated by a linear feedback shift register
  - The generator polynomial defines the period and autocorrelation function of the PN sequence
Maximum Likelihood Channel

- We will focus on channel estimation by using a known pilot signal $x$.
- The received signal is given by
  \[ y_j = \sum_{i=0}^{I-1} h_i x_{j-i} + n_j \]
- Which can also be written in matrix form
  \[ y = Xh + n \]
- With
  \[ X = \begin{bmatrix} x_0 & x_2 & x_1 \\ x_1 & x_0 & x_2 \\ x_2 & x_1 & x_0 \end{bmatrix} \]
- $y - Xh$ is a Gaussian random vector, an estimator for $h$ is given by:
  \[ \hat{h} = (X^H X)^{-1} X^H y \]
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Erfan Majeed

Sommer Semester, 2014
Lecture 9: IQ Imbalance Compensation

An analytical Model for IQ imbalance

- In a perfect world, the local oscillator’s signal for an quadrature mixer would have the following form
  \[ x_{LO}(t) = e^{-j\omega_{LO}t} = \cos(\omega_{LO}t) - j\sin(\omega_{LO}t) \]

- In reality the signal will be distorted and can be described by
  \[ x_{LO}(t) = \cos(\omega_{LO}t) - jg\sin(\omega_{LO}t + \phi) \]

- After some minor modifications we arrive at
  \[ x_{LO}(t) = K_1 e^{-j\omega_{LO}t} + K_2 e^{j\omega_{LO}t} \]

  where
  \[ K_1 = \frac{(1 + ge^{-j\phi})}{2} \quad \text{and} \quad K_2 = \frac{(1 - ge^{j\phi})}{2} \]
Mixing

• Let $z(t)$ denote the equivalent baseband signal which contains the desired signal and the image signal.

• The received signal prior to mixing $r(t)$ can then be described by
  \[
  r(t) = 2 \times \text{Re} \{ z(t) e^{j\omega_L t} \} = z(t) e^{j\omega_L t} + z^*(t) e^{-j\omega_L t}
  \]

• After mixing and subsequent lowpass filtering we arrive at
  \[
  r_{\text{IF}}(t) = K_1 z(t) + K_2 z^*(t)
  \]
Baseband Observation

- Shifting from the intermediate frequency is done via digital multiplication

\[ d(n) = K_1 s(n) + K_2 i^*(n) \]
\[ v(n) = K_2^* s(n) + K_1^* i^*(n) \]

- Or in vector notation

\[
\begin{bmatrix}
  d(n) \\
  v(n)
\end{bmatrix} =
\begin{bmatrix}
  K_1 & K_2 \\
  K_2^* & K_1^*
\end{bmatrix}
\begin{bmatrix}
  s(n) \\
  i^*(n)
\end{bmatrix}
\]
Blind Source Separation (BSS)

- Cocktail Party Example
  - Many conversations take place at the same time
  - Humans can focus on one speaker and ignore the others (BSS)
- A mathematical model
  \[ x = As \]
  - \( s \) source vector, \( x \) observed signals, \( A \) mixing matrix
- We need a separating matrix \( B \) such that
  \[ y = Bx = BAs \approx s \]
- Usually \( B \) is picked, by minimizing a cost function
Equivariant Adaptive Source Separation [2]

- Iteratively updates the separating matrix $B$
  \[ B(n + 1) = (I - \alpha(n)H(y(n)))B(n) \]

- Where
  - $I$ is the identity matrix
  - $\alpha$ is the adaption step size
  - $H$ is the matrix-valued adaption function

\[ H(y) = yy^H - I + f(y)y^H - yf(y)^H \]

- $f$ is a nonlinear function, which operates on each element of $y$, for example:

\[ f \left( \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) = \begin{bmatrix} y_1^3 \\ y_2^3 \\ y_3^3 \end{bmatrix} \]